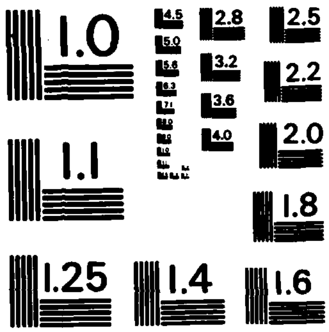
AD-A120 964 A NOTE ON SOLITARY AND CNOIDAL WAVES WITH SURFACE TENSION(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER J VANDEN-BROECK ET AL. JUL 82 MRC-TSR-2405 1/1

UNCLASSIFIED DAAG29-80-C-0041 F/G 20/4 NL

END

FILMED

DTIC


1.0
4.5
5.0
5.6
6.3
7.1
8.0
9.0
10
2.8
3.2
3.6
4.0
2.5
2.2
2.0
1.1
1.8
1.25
1.4
1.6

2

MRC Technical Summary Report #2405

A NOTE ON SOLITARY AND CNOIDAL WAVES WITH SURFACE TENSION

Jean-Marc Vanden-Broeck and M. C. Shen

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

July 1982

(Received April 7, 1982)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

A NOTE ON SOLITARY AND CNOIDAL WAVES WITH SURFACE TENSION

Jean-Marc Vanden-Broeck and M. C. Shen

Technical Summary Report #2405
July 1982

ABSTRACT

The influence of surface tension on solitary and cnoidal waves is discussed. This problem was first considered by Korteweg and de Vries [1]. Shinbrot [2] attempted to derive Korteweg and de Vries results by a formal perturbation expansion. However a part of his results is incorrect. In this note an approach different from those in [1] and [2] is used to construct a correct perturbation solution.

AMS (MOS) Subject Classification: 76B25

Key Words: solitary waves, cnoidal waves, surface tension

Work Unit Number 2 (Physical Mathematics)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant Nos. MCS800-1960 and MCS-7927062, Mod. 1.

SIGNIFICANCE AND EXPLANATION

In this note we present a perturbation solution for gravity capillary waves in water of finite depth. The calculation is valid when both the amplitude of the waves and the depth of the water are small. This problem was first considered by Korteweg and de Vries [1]. These authors showed that waves of small amplitude in relatively shallow water can be described approximately by a nonlinear differential equation. In addition Korteweg and de Vries obtained periodic solutions of this equation in closed form. They named these solutions cnoidal waves. Solitary waves are the limit of cnoidal waves as the wavelength tends to infinity.

Shinbrot [2] attempted to derive Korteweg and de Vries results by a formal perturbation expansion. However, a part of his results is incorrect, because the possibility of a depression solitary wave was excluded by him.

In this note an approach different from those in [1] and [2] is used to construct a perturbation solution which allows depression waves. The results obtained agree with those presented by Korteweg and de Vries [2].

A NOTE ON SOLITARY AND CNOIDAL WAVES WITH SURFACE TENSION

Jean-Marc Vanden-Broeck and M. C. Shen

§1. INTRODUCTION

In the present work we discuss the influence of surface tension on solitary and cnoidal waves. Solitary waves with surface tension were first considered by Korteweg and de Vries [1]. They showed that depression solitary waves exist for sufficiently large values of the surface tension. Shinbrot [2] attempted to study solitary waves with surface tension by a formal perturbation expansion. However, a part of his results is incorrect, because the possibility of a depression wave was excluded by him. In this note we use an approach different from those in [1],[2] to construct a perturbation solution which allows depression waves. In addition, the effect of surface tension on cnoidal waves is also considered. The results obtained agree with those presented by Korteweg and de Vries [1].

The problem is formulated in Section 2. The perturbation expansion for solitary waves is derived in Section 3. Cnoidal waves with surface tension are considered in Section 4.

§2. FORMULATION OF THE PROBLEM

The governing equations for the steady flow of an incompressible inviscid fluid of constant density with surface tension in reference to a moving coordinate system at a constant speed c in the x-direction are the following:

$$u^*_x + v^*_y = 0 , \tag{1}$$

$$\rho(u^* u^*_x + v^* u^*_y) = -p^*_x , \tag{2}$$

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant Nos. MCS800-1960 and MCS-7927062, Mod. 1.

$$\rho(u^* v^*_x + v^* v^*_y) = -p^*_y - \rho g \; , \tag{3}$$

subject to the boundary conditions

$$\zeta^*_t + u^* \zeta^*_x - v^* = 0 \qquad \text{at} \quad y = \zeta^*(x) \tag{4}$$

$$p^* = -T^* \zeta^*_{xx} (1 + \zeta^{*2}_x)^{3/2} \quad \text{at} \quad y = \zeta^*(x) \tag{5}$$

$$v^* = 0 \quad \text{at} \quad y = 0 \; . \tag{6}$$

Here (u^*, v^*) is the velocity, ρ the constant density, p the pressure, g the constant gravitational acceleration, $y = \zeta^*$ the equation of the free surface and T^* the constant surface tension coefficient. Following Peters and Stoker ([3]), we use x and the stream function ψ as independent variables, and transform the fluid domain to a fixed horizontal strip in the x,ψ-plane. Furthermore, it is also more convenient to use u^*, p^* and the so-called stream line function f^* as dependent variables, where f^* satisfies $\psi(x, f^*)$ = constant to define a stream line. If we introduce the nondimensional variables

$$\xi = x/h, \quad \eta = \psi/ch, \quad u = u^*/c, \quad v = v^*/c \; ,$$

$$f = f^*/h, \quad p = p^*/\rho c^2, \quad \lambda = gh/c^2, \quad T = T^*/\rho g h^2 \; .$$

where h is the equilibrium fluid depth, (1) to (6) become

$$u_\xi = -f_\eta p_\xi + f_\xi p_\eta \; , \quad -\infty < \xi < \infty \; , \tag{7}$$

$$u f_{\xi\xi} + u_\xi f_\xi = -\lambda f_\eta - p_\eta \; , \quad 0 < \eta < 1 \; , \tag{8}$$

$$u f_\eta = 1 \; , \tag{9}$$

$$f = 0, \quad \text{at} \quad \eta = 0, \quad -\infty < \xi < \infty \; , \tag{10}$$

$$p = -T f_{\xi\xi} / (1 + f_\xi^2)^{3/2} \quad \text{at} \quad \eta = 1, \quad -\infty < \xi < \infty \; . \tag{11}$$

Assume λ is close to some critical value ℓ and let

$$\varepsilon = \ell - \lambda, \quad \sigma = \xi(\varepsilon)^{1/2} \; , \tag{12}$$

then (7) to (11) in terms of σ become

$$u_\sigma = -f_\eta p_\sigma + f_\sigma p_\eta \; , \tag{13}$$

$$\varepsilon(uf_{\sigma\sigma} + u_\sigma f_\sigma) = \varepsilon f_\eta - \ell f_\eta - p_\eta \tag{14}$$

$$uf_\eta = 1 \ , \tag{15}$$

$$f = 0, \quad \text{at } \eta = 0 \ , \tag{16}$$

$$p = -T\varepsilon f_{\sigma\sigma}/(1 + \varepsilon f_\sigma^2)^{3/2} \quad \text{at } \eta = 1 \ . \tag{17}$$

§3. SOLITARY WAVE SOLUTIONS

To solve (13) to (17), we assume

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots \tag{18}$$

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots \tag{19}$$

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots \tag{20}$$

where

$$u_0 = 1, \quad f_0 = \eta, \quad p_0 = -\ell(\eta - 1) \ . \tag{21}$$

Substitution of (18) to (21) in (13) to (17) will yield a sequence of equations and boundary conditions for the successive approximations. The equations for the first approximations are

$$u_{1\sigma} = - p_{1\sigma} - \ell f_{1\sigma} \ , \tag{22}$$

$$1 - \ell f_{1\eta} - p_{1\eta} = 0 \ , \tag{23}$$

$$u_1 + f_{1\eta} = 0 \ , \tag{24}$$

$$f_1(\sigma,0) = 0 \ , \tag{25}$$

$$p_1(\sigma,0) = 0 \ . \tag{26}$$

From (21) to (22), we have

$$f_{1\sigma\eta\eta} = 0 \ . \tag{27}$$

By integration and making use of (25), we obtain

$$f_{1\sigma} = a_1'(\sigma)\eta \ ,$$

where $a_1(\sigma)$ is to be determined. By integration again,

$$f_1 = a_1(\sigma)\eta + b_1(\eta) .$$

Here we assume that $a_1(\sigma) \to 0$ as $\sigma \to \infty$ or $a_1(\sigma)$ is a periodic function with period L and $\int_0^L a_1(\sigma)d\sigma = 0$. It follows that

$$f_1 = a_1(\sigma)\eta . \tag{28}$$

From (23), (24) and (26), we have

$$u_1 = -a_1(\sigma) , \tag{29}$$

$$p_1 = [1 - \ell a_1(\sigma)](\eta - 1) . \tag{30}$$

By substituting (28) to (30) in (22), it is found that

$$a_1'(\sigma)(\ell - 1) = 0 . \tag{31}$$

If $a_1'(\sigma) \not\equiv 0$, then

$$\ell = 1 . \tag{32}$$

The equations for the second approximations are

$$u_{2\sigma} = -f_{2\sigma} - p_{2\sigma} + a_1'(\sigma)\eta - a_1(\sigma)a_1'(\sigma) , \tag{33}$$

$$a_1''(\sigma)\eta = a_1(\sigma) - f_{2\eta} - p_{2\eta} , \tag{34}$$

$$f_{2\eta} + u_2 - a_1^2(\sigma) = 0 , \tag{35}$$

$$f_2(\sigma,0) = 0 , \tag{36}$$

$$p_2 = -Ta_1''(\sigma) \quad \text{at} \quad \eta = 1 , \tag{37}$$

where (28) to (30) and (32) have been used. As before we find that

$$f_2 = a_2(\sigma)\eta - a_1''(\sigma)\eta^3/6 , \tag{38}$$

$$u_2 = -a_2(\sigma) + a_1^2(\sigma) + a_1''(\sigma)\eta^2/2 , \tag{39}$$

$$p_2 = -a_2(\sigma)(\eta - 1) + a_1(\sigma)(\eta - 1) + a_1''(\sigma)(-T + \eta^3/6 - \eta^2/2 + 1/3) . \tag{40}$$

Substitution of (38) and (40) in (33) yields

$$(1/3 - T)a_1'''(\sigma) = a_1'(\sigma) - 3a_1(\sigma)a_1'(\sigma) . \tag{41}$$

If we impose the condition $a_1'(\sigma)$, $a_1''(\sigma) \to 0$ as $\sigma \to \infty$ and $a_1'(0) = 0$, then by integration we obtain

$$a_1(\sigma) = \mathrm{sech}^2(\sigma/2)(1/3 - T)^{-1/2} \tag{42}$$

It follows from (19), (21), (28) and (42) that

$$f^* \sim h + h(1-\lambda)\mathrm{sech}^2(x/2h)[(1-\lambda)/(1/3 - T)]^{1/2} . \tag{43}$$

If $T < 1/3$ and $\lambda < 1$, there is a solitary wave of elevation. In this case, $c^2 > gh$ and the solitary wave moves at a supercritical speed. If $T > 1/3$ and $\lambda > 1$, there is a solitary wave of depression moving at a subcritical speed units $c^2 < gh$. We may define

$$a = |h(1-\lambda)| ,$$

and (43) in terms of a becomes

$$f^* \sim h \pm a\ \mathrm{sech}^2(x/2h)[\pm a/(1/3 - T)]^{1/2} . \tag{44}$$

If we consider a periodic solution of (41), then by integration

$$(1/3 - T)[a_1'(\sigma)]^2 = -a_1^3(\sigma) + a_1^2(\sigma) + c_1 a_1(\sigma) + c_2 , \tag{45}$$

where c_1, c_2 are two arbitrary constants. The periodic solutions are called cnoidal waves and will be discussed in the next section. Finally if $T = 1/3$, then (41) becomes

$$a_1'(\sigma)(1 - 3a_1(\sigma)) = 0 , \tag{46}$$

and the only possible solution is $a_1(\sigma) = 0$.

§4. CNOIDAL WAVE SOLUTION

For a bounded, periodic solution of (45), it is necessary that the right hand side of (45) must have three real simple zeros, and the solution will occillate periodically between two of the zeros. Therefore, we rewrite (45) in the form

$$(1/3 - T)[a_1'(\sigma)]^2 = [h_1 - a_1(\sigma)][a_1(\sigma) - h_2][a_1(\sigma) - \gamma] , \tag{47}$$

where h_1, h_2 are the extremum values of $a_1(\sigma)$ and

$$\gamma = 1 - h_2 - h_1 . \tag{48}$$

Without loss of generality we shall assume that h_1 and h_2 satisfy the inequality

$$\left(\frac{1}{3} - T\right)(1 - \lambda)(h_1 - h_2) > 0$$

Following Korteweg and de Vries [1] we introduce the parameter X and write

$$a_1(\sigma) = h_1\cos^2 X + h_2\sin^2 X \ . \tag{49}$$

Substituting (49) into (47), we obtain

$$\beta \frac{dX}{d\sigma} = (1 - K^2\sin^2 X)^{1/2} \ . \tag{50}$$

Here β and K^2 are defined by

$$\beta = \left(\frac{4}{3}\,\frac{1 - 3T}{h_1 - \gamma}\right)^{1/2} , \tag{51}$$

$$K^2 = \frac{h_1 - h_2}{h_1 - \gamma} \ . \tag{52}$$

We take the origin of σ at a point where $a_1(\sigma)$ attains the value h_1. Thus integrating (50) we obtain

$$\sigma = \beta \int_0^X \frac{dX}{(1 - K^2\sin^2 X)^{1/2}} = \beta F(X,K) \ . \tag{53}$$

We can rewrite (53) in terms of Jacobian elliptic functions as

$$\cos X = \mathrm{cn}\,\frac{\sigma}{\beta} \ . \tag{54}$$

Substituting (56) into (49) we have

$$a_1(\sigma) = h_2 + (h_1 - h_2)\mathrm{cn}^2\,\frac{\sigma}{\beta} \ . \tag{55}$$

It follows from (19), (21), (28) and (55) that

$$f^* \sim h + (1-\lambda)hh_2 + (1-\lambda)h(h_1 - h_2)\mathrm{cn}^2\left(\frac{x}{2h}\right)\left[(h_1 - \gamma)(1-\lambda)/(1/3 - T)\right]^{1/2} \tag{56}$$

The wavelength L of the wave is given by

$$L = 4h\left[\frac{1 - 3T}{3(h_1 - \gamma)(1 - \lambda)}\right]^{1/2} F_1(K) , \tag{57}$$

where

$$F_1(K) = \int_0^{\pi/2} (1 - K^2 \sin^2 X)^{-1/2} dX . \tag{58}$$

The determination of the solutin is completed by imposing the condition $\int_0^L a_1(\sigma)d\sigma = 0$ where the integral is over a wavelength. Using (49), (50) and (52) we have

$$0 = 2\beta \int_0^{\pi/2} \frac{h_1 \cos^2 X + h_2 \sin^2 X}{(1 - K^2 \sin^2 X)^{1/2}} dX$$

$$= 2\beta [\gamma F_1(K) + (h_1 - \gamma)E_1(K)] . \tag{59}$$

Here

$$E_1(K) = \int_0^{\pi/2} (1 - K^2 \sin^2 X)^{1/2} dX .$$

Hence

$$\gamma F_1(K) + (h_1 - \gamma)E_1(K) = 0 . \tag{60}$$

The eight quantities λ, h_1, h_2, γ, K, L, T and β are related by the five equations (48), (51), (52), (57) and (60). If for instance λ, L and T are given the others can be found.

If $L = \infty$, relations (57), (52) (60) and (48) yield $K = 1$, $\gamma = h_2 = 0$ and $h_1 = 1$. The cnoidal wave reduces then to the solitary wave considered in Section 3.

For infinitesimal amplitude, equation (41) reduces to

$$(1/3 - T)a_1''(\sigma) = a_1'(\sigma) . \tag{61}$$

The solution is

$$a_1(\sigma) = h_1 \cos(T - 1/3)^{-1/2}\sigma \tag{62}$$

It follows from (19), (21), (28) and (62) that

$$f^* = h + h_1(1 - \lambda)\cos\frac{x}{h}[(1 - \lambda)/(T - 1/3)]^{1/2} . \tag{63}$$

If we introduce

$$\kappa = [(1 - \lambda)/(T - 1/3)]^{1/2}/h , \tag{64}$$

we have

$$f^* = h + hh_1(1 - \lambda)\cos \kappa x . \tag{65}$$

Relation (64) can be rewritten as

$$\frac{gh}{c^2} = 1 - \kappa^2 h^2 (T - 1/3) . \tag{66}$$

The dispersion relation of linear sine waves is given by

$$c^2 = \frac{g}{\kappa}\tanh \kappa h[1 + \frac{T^*}{\rho g}\kappa^2] . \tag{67}$$

In the limit as $\kappa h \to 0$, (67) may be expanded as

$$\frac{c^2}{gh} \sim 1 + (T - 1/3)\kappa^2 h^2 + o(\kappa^2 h^2) . \tag{68}$$

A comparison of (67) and (68) indicates that the cnoidal wave solution overlaps the classical linear theory in the limit as the amplitude of the waves tends to zero.

REFERENCES

[1] P. J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, Phil. Mag., 39, 422-443 (1895).

[2] M. Shinbrot, The solitary wave with surface tension, Quat. App. Math., 39, 287-291 (1981).

[3] A. S. Peters and J. J. Stoker, Solitary waves in liquids having non-constant density, Comm. Pure App. Math., 13, 115-164 (1960).

JMV/MCS/ed

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 2405

2. GOVT ACCESSION NO.: AD-A120 964

3. RECIPIENT'S CATALOG NUMBER:

4. TITLE (and Subtitle): A NOTE ON SOLITARY AND CNOIDAL WAVES WITH SURFACE TENSION

5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period

6. PERFORMING ORG. REPORT NUMBER:

7. AUTHOR(s): Jean-Marc Vanden-Broeck and M. C. Shen

8. CONTRACT OR GRANT NUMBER(s): MCS800-1960; DAAG29-80-C-0041; MCS-7927062, Mod. 1

9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: Work Unit Number 2 - Physical Mathematics

11. CONTROLLING OFFICE NAME AND ADDRESS: See Item 18 below

12. REPORT DATE: July 1982

13. NUMBER OF PAGES: 9

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):

15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

solitary waves, cnoidal waves, surface tension

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The influence of surface tension on solitary and cnoidal waves is discussed. This problem was first considered by Korteweg and de Vries, ~~[1]~~. Shinbrot [2] attempted to derive Korteweg and de Vries results by a formal perturbation expansion. However, a part of his results is incorrect. In this note an approach different from those ~~in~~ [1] and [2] is used to construct a correct perturbation solution.